

Göteborg ITP 95-21

September 1995

Revised: January 1996

hep-th/9509111

# Gauge fixing, co-BRST and time evolution in QED

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## Abstract

We consider BRST-invariant inner product states for quantum electrodynamics constructed from trivial BRST-invariant states and a gauge regulator. The trivial states are products of matter and ghost states and are annihilated by hermitian operators. The co-BRST operator and some further gauge-fixing regulators are found. The relation between gauge fixing and time evolution of both the trivial and the inner product states is discussed.

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# 1 Introduction

As discussed in [1] for models with finite number of degrees of freedom the physical inner product states can in general be obtained by means of a gauge fixing regulator from two simple solutions of the fundamental BRST-equation:

$$Q | \varphi \rangle = 0 \quad (1.1)$$

where  $Q$  is the nilpotent hermitian BRST charge. These simple solutions to equation (1.1) are products of matter and ghost states:  $| \varphi_l \rangle \equiv | \text{matter} \rangle | \text{ghost} \rangle, l = 1, 2$ . In models with finite number of degrees of freedom the  $| \varphi_l \rangle$  states were shown not to be well defined inner product states by themselves in [1]. They could be used however to obtain well defined inner product BRST-singlet states (denoted by  $| s_l \rangle$ ) by acting on them with some gauge-fixing regulators:

$$| s_l \rangle = e^{\gamma_l K_l} | \varphi_l \rangle, \quad l = 1, 2 \quad (1.2)$$

where  $K_l = [\rho_l, Q]$ ,  $\rho_l$  being a fermionic hermitian gauge-fixing operator and  $\gamma_l$  is a real coefficient [2]. In [3] it was shown that the same singlets can be obtained using linear combinations of the previous operators  $\alpha_l K_1 + \beta_l K_2$  if a certain relation between  $\alpha_l, \beta_l$  and  $\gamma_l$  was satisfied. With some abuse of the language we sometimes call the operators  $K_l$  also gauge-fixing operators, but this will never lead to confusions about which operators we mean. For non-abelian models the singlets  $| s_l \rangle$  can no longer be written as products of purely matter and ghost states.

The singlets are annihilated by a set of non-hermitian operators. By the hermitian conjugates of these operators one can build a basis for the unphysical inner product states. One can form linear combinations of these inner product states to get an orthogonal basis on the inner product state space. The elements of this orthogonal basis are denoted by  $| \xi \rangle$ . Half of the  $| \xi \rangle$  states have positive norm and half of them negative. The hermitian metric operator ( $\hat{\eta}$ ) which gives

$$\langle \xi | \hat{\eta} | \xi \rangle > 0, \quad \forall | \xi \rangle \quad (1.3)$$

defines the co-BRST operator by

$${}^*Q = \eta^{-1} Q \eta \quad (1.4)$$

The physical space is completely determined by the BRST and the co-BRST operators [4]. One can always chose the metric operator to be idempotent ( $\eta^2 = 1$ ) and then the co-BRST operator becomes  ${}^*Q = \eta Q \eta$ . Physical states are those that are annihilated by both these operators.

One of the purposes with this paper is to find the appropriate generalizations of the above results to the case of infinitely many degrees of freedom and because of this it is natural to consider first free electrodynamics. The hope is to learn facts about the regulation procedure and the co-BRST also valid for more general field theories. The two simple BRST-invariant states (denoted also here by  $|\varphi_l\rangle$ ) are also here shown not to be inner product states. By (1.2) one can find the corresponding inner product states ( $|s\rangle$ ). As it will be shown later on, every inner product state can in this case be obtained from both the  $|\varphi_l\rangle$  states since

$$\exp(\gamma K_1) |\varphi_1\rangle = \exp\left(\frac{1}{\gamma} K_2\right) |\varphi_2\rangle \quad (1.5)$$

In this sense the two trivial states  $|\varphi_1\rangle$  and  $|\varphi_2\rangle$  are completely equivalent. In non-Abelian theories the corresponding states have to be treated somewhat differently. (See [3] for a model with finite number of degrees of freedom.) Using the method mentioned before (equations (1.3 and (1.4)) one can find the co-BRST operator, in fact a whole class of them: one for each value of  $\gamma$ . The  $|s\rangle$  state found in equation (1.2) is the physical vacuum. All the other physical states can be built on it by physical creation operators. From the freedom in the choice of the co-BRST charge one realises that there is a one-parameter freedom in the choice of the vacuum state.

Since in electrodynamics we have an explicit non-vanishing Hamiltonian one is also interested in the time-evolution of both the trivial BRST-invariant  $|\varphi_l\rangle$  and the inner product  $|s\rangle$  states. It seems that the gauge fixing regulators are related to time evolution. The Hamiltonian of free electrodynamics can be written as the sum of a BRST-closed and a BRST-exact term. A formal similarity between the time evolution operator on the equation of motion level and the gauge-fixing regulator leads us to consider an operator of the form  $\exp(H_{ph} + [Q, \rho])$  and decompose it. We are led to the conclusion that the vacuum singlets obtained before are evolved in time as expected by a physical Hamiltonian which only depends on the orthogonal components of the electric field and the magnetic

field. The non-physical part of the Hamiltonian is a simple combination of the gauge-fixing operators  $K_l$ . The time-evolution operator built on this Hamiltonian in an imaginary-time formalism is a generalized gauge-regulator. The trivial  $|\varphi_l\rangle$  states are exactly defined by the operators (four for each state) annihilating them. One of these operators for each trivial state is a gauge-fixing operator, sometimes called gauge-slicing. As it is shown in Section 4 there exists a gauge-slicing such that the state it defines evolves under an imaginary-time evolution operator equivalent to the gauge regulator itself.

There exist several features that differ in this paper as compared to previous works. A fundamental analysis of quantum electrodynamics and Yang-Mills theories was given by Kugo and Ojima in [5]. The main difference between [5] and the present paper is the fact that in [5] one uses a perturbative Fock-space approach. One builds all states on a vacuum that is itself an inner product state. The present paper uses a non-perturbative approach, the fundamental states being eigen-states of the connections (scalar and vector potentials) and the electric fields. We look for BRST-invariant states of the form  $|matter\rangle |ghost\rangle$ . This leads us as proven in [1], [2] to states (including the vacuum state) that are not well defined inner product states. Since this situation occurs quite generally it can be instructive to first analyze a simple case like quantum electrodynamics. In the physical state space as defined in [5] it is the quartet mechanism that guarantees that unphysical states always appear in zero-norm combinations, thus not affecting any physical inner products. In the present paper we define the physical inner product states as the one being annihilated by both the BRST and the co-BRST operators. This definition does not leave room for any non-physical states in the physical state space and not even zero-norm states are left.

In Section 2 we first introduce the formalism which is built on the eigen-vectors of the connection (the vector and scalar potential). This basis is very convenient since every operator in the Hamiltonian formulation corresponds to either the connections or to their momenta. For more complicated models the non-perturbative nature of these eigen-vectors can also be an advantage. We explicitly prove that the trivial BRST-invariant states, that is solutions to eq.(1.1) denoted by  $|\varphi_1\rangle$  and  $|\varphi_2\rangle$ , for free electrodynamics are not inner product states. More exactly it is shown that  $\langle\varphi_l|\varphi_l\rangle=0\infty$ ,  $l=1,2$ .

In Section 3 we find the well defined inner product states using the prescription given

by Batalin and Marnelius in [1]. We show that these states are also normed to unity if the eigen-value of the time-like component of the connection (the scalar potential) operator is imaginary. It is shown that the BRST and co-BRST operators together completely define the singlet vacuum state, as expected from [4].

The two gauge fixing operators  $K_l, l = 1, 2$  are members of an  $SL(2, R)$  algebra, thus one can use (whenever convenient) linear combinations of them instead of any of them. These details are discussed in Section 4 and the conclusions we arrive at are essentially the same as in the case of finite number of degrees of freedom [3].

The second part of Section 4 deals with the time evolution of the  $|s\rangle$  and the  $|\varphi_l\rangle$  states. It contains the decomposition of the Hamiltonian into a physical part and a gauge-fixing term. We show a way how one can understand this decomposition in an imaginary-time formalism. We also comment on a possible interpretation in the real time case related to the path-integral formalism. The Appendix contains some details of the decomposition of the unphysical Hamiltonian into a combination of the gauge-fixing operators  $K_l$  and another unphysical operator.

## 2 The BRST-invariant states

The formalism used throughout this paper is built on the eigenvectors of the connection ( $A^\mu(x)$ ) and the fermionic ghost and anti-ghost ( $\eta(x)$  and  $\bar{\eta}(x)$ ) operators:

$$|A\rangle = \prod_{\vec{x}} \prod_{\mu} |A^\mu(\vec{x})\rangle, |\eta\rangle = \prod_{\vec{x}} |\eta(\vec{x})\rangle, |\bar{\eta}\rangle = \prod_{\vec{x}} |\bar{\eta}(\vec{x})\rangle \quad (2.1)$$

where

$$\hat{A}^i(x) |A^i(\vec{x})\rangle = A^i(x) |A^i(\vec{x})\rangle \quad (2.2)$$

$$\hat{A}^0(x) |A^0(\vec{x})\rangle = iA^0(x) |A^0(\vec{x})\rangle \quad (2.3)$$

$$\hat{\eta}(x) |\eta(\vec{x})\rangle = \eta(x) |\eta(\vec{x})\rangle \quad (2.4)$$

$$\hat{\bar{\eta}}(x) |\bar{\eta}(\vec{x})\rangle = \bar{\eta}(x) |\bar{\eta}(\vec{x})\rangle \quad (2.5)$$

There are two reasons for this choice. One of them is that the connections, the ghosts and their momenta are the fundamental variables of the theory both in the Lagrangean and the Hamiltonian formalism. In the Hamiltonian formalism all the operators, including

the BRST-operator, are expressed as functionals of the operators that correspond to the fundamental variables. The other reason is the non-perturbative nature of such eigenstates in a more general context. As there exist several areas (like quantum gravity (see e.g. [6])) where perturbative methods do not lead to renormalizable theories one is lead to study some intrinsically non-perturbative formalisms ([7]).

The eigenvalue of the time component of the connection is imaginary (2.3) because this is necessary for the normalizability of the states. Vectors of this kind were first introduced by Pauli [8]. For a detailed analysis see e.g.[9]. The connection eigen-states (2.2 and 2.3) form a complete and orthonormal basis:

$$\langle A' | A \rangle = \prod_x \delta^3 \left( A'^i(x) - A^i(x) \right) \delta \left( A'^0(x) + A^0(x) \right) \quad (2.6)$$

The last term follows from the imaginary eigenvalue of  $A^0$ :

$$\langle A' | \hat{A}^0 | A \rangle = -iA'^0(x)\langle A' | A \rangle = \langle A' | A \rangle iA^0(x) \quad (2.7)$$

The eigenvalue equations of the ghost operators are defined in the same way as the equations of the matter operators (in equations (2.4) and (2.5)) but there exist certain differences due to the fermionic nature of the ghosts. For some details about eigenvalue equations of fermionic operators see e.g.[10]. One of their important properties that we have to keep in mind is the fact that the  $\delta$ -function of a fermionic variable is proportional to the variable itself:

$$\delta(\eta - \eta') = (\eta - \eta') \quad (2.8)$$

This relation results in the following remarkable property of the ghost inner product states:

$$\langle \eta | \eta' \rangle \propto \delta(\eta - \eta') = -i(\eta - \eta') \quad (2.9)$$

which vanishes for  $|\eta\rangle = |\eta'\rangle$ . A general wave functional expressed in the basis (2.1) is the Schrödinger representation:

$$\Phi[A, \eta, \bar{\eta}] = \langle A, \eta, \bar{\eta} | \varphi \rangle \quad (2.10)$$

(Throughout this paper both the wave-functional and the ket forms are going to be used depending on which of them is more convenient.)

The physical states in a theory are expressed by the cohomology classes of the BRST-operator i.e. classes of states that are eliminated by it but which cannot be written as the BRST-operator acting on any other state. When solving the equation  $Q | \phi \rangle = 0$  one finds two different classes of solutions built on different ghost vacua. These states are not inner product states as it was proved in [1] for systems with finite number of degrees of freedom. As we shall see in this section the situation is the same in the case of electrodynamics too.

An important assumption in [11] was that one can always write the wave functionals corresponding to physical states as products of a matter and a ghost part:

$$\Phi[A, \eta, \bar{\eta}] = \langle A | \varphi \rangle_m \langle \eta, \bar{\eta} | \varphi \rangle_{gh} \equiv \Phi_m[A] \Phi_{gh}[\eta, \bar{\eta}] \quad (2.11)$$

This decomposition, together with the completeness of the  $| A \rangle$  basis leads us to the formal inner product for the matter part

$$\langle \varphi | \Psi \rangle = \int \mathcal{D}A \Phi_m^*[A] \Psi_m[A] \quad (2.12)$$

where  $\mathcal{D}A = \prod_{\vec{x}} \prod_{\mu} dA^{\mu}(\vec{x})$ . The ghost part does not yield a well defined inner product because of the properties described in eq.(2.9). Another natural and much more geometrical way to describe BRST-quantization would be to work on the space of all the connections. The BRST-operator acts in this space as an exterior derivation operator. (See e.g. [12], [13] and for an example [14].)

The Lagrangian we are going to use is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_{\mu} A^{\mu})^2 - i \partial_{\mu} \bar{\eta} \partial^{\mu} \eta \quad (2.13)$$

The imaginary unit ( $i$ ) appears because we demand that both the Lagrangian and the ghost variables are real. One could of course redefine the ghosts to be complex conjugated to the anti-ghosts and then the  $i$  would disappear. The reality of the ghost variables causes their momenta to be purely imaginary. In the Hamilton formalism the non-zero commutators between the fundamental operators are as follows:

$$[A^{\mu}(x), E^{\nu}(y)]_{x^0=y^0} = i g^{\mu\nu} \delta^3(\vec{x} - \vec{y}) \quad (2.14)$$

$$[\eta(x), \mathcal{P}(y)]_{x^0=y^0} = i \delta^3(\vec{x} - \vec{y}) \quad (2.15)$$

$$[\bar{\eta}(x), \bar{\mathcal{P}}(y)]_{x^0=y^0} = i \delta^3(\vec{x} - \vec{y}) \quad (2.16)$$

where  $g^{\mu\nu}$  is the Minkowski metric  $\text{Diag}[-1, 1, 1, 1]$ . The BRST charge of free electrodynamics in the phase space [15] is:

$$Q = \int d^3x \left( \partial_i E^i(x) \eta(x) - i E^0(x) \bar{\mathcal{P}}(x) \right) \quad (2.17)$$

This somewhat unconventional form of the term containing the anti-ghost momenta multiplied by the imaginary unit insures the hermiticity of the BRST-charge. This BRST-charge is only nilpotent using the equations of motion. One also notices that this Lagrangian is anti-BRST invariant too, the anti-BRST charge in abelian models being of the same form as the BRST-charge with the ghost and anti-ghost variables (and momenta) interchanged:

$$\bar{Q} = \int d^3x [\partial_i E^i(x) \bar{\eta}(x) + i E^0(x) \mathcal{P}(x)] \quad (2.18)$$

For more details about the anti-BRST charge and its role in quantizing gauge theories see e.g.[16]. As a first step in searching for the physical states we are looking for the BRST-invariant and anti-BRST invariant ones, i.e. those satisfying:

$$Q\Phi[A, \eta, \bar{\eta}] = 0 \quad (2.19)$$

$$\bar{Q}\Phi[A, \eta, \bar{\eta}] = 0 \quad (2.20)$$

Using the decomposition of the wave-functional into a matter and a ghost part from eq.(2.11) one obtains the two classes of solutions mentioned in the introduction. The **first** class of solutions are those states that satisfy:

$$E^0(\vec{x})\Phi_{1m}[A] = \eta(\vec{x})\Phi_{1gh}[\eta, \bar{\eta}] = \bar{\eta}(\vec{x})\Phi_{1gh}[\eta, \bar{\eta}] = 0 \quad (2.21)$$

The **second** class is given by:

$$\partial_i E^i(\vec{x})\Phi_{2m}[A] = \bar{\mathcal{P}}(\vec{x})\Phi_{2gh}[\eta, \bar{\eta}] = \mathcal{P}(\vec{x})\Phi_{2gh}[\eta, \bar{\eta}] = 0 \quad (2.22)$$

There are two more conditions, one for each state, that can be consistently imposed on these states and they are gauge-fixing conditions of the form [1]

$$-\frac{\partial_i}{\Delta} A^i(x) \mid \varphi_1 \rangle = 0 \quad (2.23)$$

$$A^0(x) \mid \varphi_2 \rangle = 0 \quad (2.24)$$



The matter part of the **first** equation (2.21) only tells us that these states do not depend on  $A^0$ . The ghost part of (2.21) gives the ghost vacuum as solution. That is the eigenvalues of both the ghost and the anti-ghost operators vanish all over the space. It follows then that the states corresponding to the solution of the eq.(2.21) are of the form

$$| \varphi_1 \rangle = | \varphi[\vec{A}] \rangle \otimes \int \mathcal{D}A^0 | A^0 \rangle \otimes | 0 \rangle_{\eta\bar{\eta}} \quad (2.25)$$

These states are obviously not well defined inner product states.  $\langle {}_1\varphi | \varphi_1 \rangle$  results in  $\infty 0$  in every space point, the infinity comes from the integral by  $dA^0$  and the zero from the ghost part (see eq.2.9).

The **second** class of solutions to the BRST- and anti-BRST equations is given by eq.(2.22). The ghost parts of the equation only tell us that the ghost part of the state does not depend on the ghost  $\eta$  and the anti-ghost  $\bar{\eta}$ . The ket corresponding to this solution can then be written as  $\int \mathcal{D}\eta | \eta \rangle \otimes \int \mathcal{D}\bar{\eta} | \bar{\eta} \rangle$ . One can also understand these states as the ghost-momentum-vacuum. The matter part of equation (2.22) can easily be solved if one goes over to the conjugated  $E$  representation:

$$\Phi_m[A] = \int \mathcal{D}E \langle A | E \rangle \langle E | \varphi \rangle = \int \mathcal{D}E e^{\frac{i}{\hbar} \int d^3x A_\mu(x) E^\mu(x)} \Phi[E] \quad (2.26)$$

If  $\Phi_m[A]$  is to satisfy (2.22)  $\Phi[E]$  has to be of the form  $\prod_x \delta(\partial_i E^i(x)) \tilde{\Phi}[E]$ . By writing the naive integral measure as:  $dE^0 d^2 E^\perp dE^\parallel$  and using the fact that:

$$E^\parallel = \frac{\partial_i}{\sqrt{-\Delta}} E^i$$

one obtains:

$$\Phi_m[A] = \int \mathcal{D}E^0 \mathcal{D}^2 E^\perp \frac{1}{\sqrt{-\Delta}} \tilde{\Phi}(E^0, E^\perp, E^\parallel = 0) e^{\frac{i}{\hbar} \int d^3x (A_0 E^0 + A^\perp E^\perp)} \quad (2.27)$$

The  $\sqrt{-\Delta}$  operator is a well-defined real and positive operator in the momentum space representation. Since the space-time metric is  $g_{\mu\nu} = \text{Diag}(-1, 1, 1, 1)$   $(-\Delta)$  corresponds to  $k_i k^i$  in the momentum space. The obvious problem with this solution is the same as in case 1: it is not a well defined inner product state. One can see this by taking the naive inner product between two states of the form  $| \Phi \rangle = \int \mathcal{D}A \Phi_m[A] | A \rangle \int d\eta | \eta \rangle \int d\bar{\eta} | \bar{\eta} \rangle$ . The result is again an ambiguous  $\infty 0$ , where the infinity comes from the integral over the parallel component of the connection and the zero comes from the (anti-)ghost part

of the integral following from eq.(2.9). The same result can be obtained even without introducing the  $|E\rangle$  kets. One can simply chose another base for the connection and write the integral measure decomposed into a time-like, an orthogonal and a parallel part. The matter part of the state becomes:

$$|\Phi\rangle_m = \prod_x \int dA^0 d^2 A^\perp dA^\parallel \Phi[A] |A^0\rangle |A^\perp\rangle |A^\parallel\rangle \quad (2.28)$$

The constraint expressed in (2.22) means essentially that  $\Phi[A]$  does not depend on the parallel component of  $A$ : it has to be a function of e.g.  $A_i(g^{ij} - \frac{\partial^i \partial^j}{\Delta})A_j = A^\perp A^\perp$ . For a more detailed description see [17]. So neither of the two trivial solutions to the BRST equation (1.1) is an inner product state.

### 3 Inner Product States

To summarize what has been said up till now: the fundamental equation  $(Q|\varphi\rangle = 0)$  has two trivial solutions,  $|\varphi_1\rangle$  and  $|\varphi_2\rangle$  defined by the relations (2.21) and (2.23) resp. (2.22) and (2.24) or expressed in the ket-notation:

$$\partial_i A^i(x) |\varphi_1\rangle = E^0(x) |\varphi_1\rangle = \eta(x) |\varphi_1\rangle = \bar{\eta}(x) |\varphi_1\rangle = 0 \quad (3.1)$$

$$A^0(x) |\varphi_2\rangle = \partial_i E^i(x) |\varphi_2\rangle = \mathcal{P}(x) |\varphi_2\rangle = \bar{\mathcal{P}}(x) |\varphi_2\rangle = 0 \quad (3.2)$$

None of the solutions of these conditions is an inner product state. Since physical states have to be inner product states we have to find such states built on the solutions we obtained. The inner product states have to be also BRST-invariant.

The way this problem is dealt with in models with finite degrees of freedom [1] is that one acts with gauge fixing regulators on the two  $|\varphi_i\rangle$  states. These gauge fixing regulators are the exponentials of commutators between the BRST operator and a gauge-fixing condition so the BRST invariance of the new states is automatically guaranteed.

In the case of free electrodynamics the inner product states will be of the form

$$|s_l\rangle = e^{\gamma^l[\rho_l, Q]} |\varphi_l\rangle, \quad l = 1, 2 \quad (3.3)$$

The gauge-fixing operators in this case are:

$$\rho_1 = -i \int d^3x \frac{1}{\sqrt{-\Delta}} A^0(x) \mathcal{P}(x) \quad (3.4)$$

$$\rho_2 = \int d^3x \frac{\partial_i}{\sqrt{-\Delta}} A^i(x) \bar{\eta}(x) = \int d^3x A^\parallel(x) \bar{\eta}(x) \quad (3.5)$$

and  $\gamma_l$  are just constants. (There is no summation over the indices  $l$ .) The operators involving  $\sqrt{-\Delta}$  are defined as in the previous section by their correspondents in the momentum space.  $A^\parallel(x)$  is the parallel component of the gauge-field  $A^\mu(x)$ . Its conjugated momentum is:

$$E^\parallel(x) = \frac{\partial_i}{\sqrt{-\Delta}} E^i(x) \quad (3.6)$$

and the equal-time commutation relation between them is

$$[A^\parallel(x), E^\parallel(y)] = i\delta^3(\vec{x} - \vec{y}) \quad (3.7)$$

From here one easily obtains:

$$K_1 \equiv [Q, \rho_1] = \int d^3x \left( A^0(x) E^\parallel(x) + i\mathcal{P}(x) \frac{1}{\sqrt{-\Delta}} \bar{\mathcal{P}}(x) \right) \quad (3.8)$$

$$K_2 \equiv [Q, \rho_2] = \int d^3x \left( E^0(x) A^\parallel(x) + i\bar{\eta}(x) \sqrt{-\Delta} \eta(x) \right) \quad (3.9)$$

The operators that annihilate the singlet states obtained in this way are given by

$$D'_l = e^{\gamma_l K_l} D_l e^{-\gamma_l K_l} \quad (3.10)$$

where  $D_l$  stands for all the operators annihilating  $|\varphi_l\rangle$ . Again there is no summation over  $l$  included. Since from  $\partial_i A^i |\varphi_1\rangle = 0$  follows that  $A^\parallel |\varphi_1\rangle = 0$  one can use  $A^\parallel$  in eq.(3.1). The situation is of course the same for the  $\partial_i E^i$  in the case of the second state: one can replace it by  $E^\parallel$ . The conditions which  $|s_1\rangle$  satisfies are:

$$\begin{aligned} (A^\parallel(x) - i\gamma_1 A^0(x)) |s_1\rangle &= (E^0(x) - i\gamma_1 E^\parallel(x)) |s_1\rangle = \\ &= \left( \eta(x) + \gamma_1 \frac{1}{\sqrt{-\Delta}} \bar{\mathcal{P}}(x) \right) |s_1\rangle = \left( \bar{\eta}(x) - \gamma_1 \frac{1}{\sqrt{-\Delta}} \mathcal{P}(x) \right) |s_1\rangle = 0 \end{aligned} \quad (3.11)$$

while the conditions on  $|s_2\rangle$  are:

$$\begin{aligned} (A^0(x) + i\gamma_2 A^\parallel(x)) |s_2\rangle &= (E^\parallel(x) + i\gamma_2 E^0(x)) |s_2\rangle = \\ &= (\mathcal{P}(x) - \gamma_2 \sqrt{-\Delta} \bar{\eta}(x)) |s_2\rangle = (\bar{\mathcal{P}}(x) - \gamma_2 \sqrt{-\Delta} \eta(x)) |s_2\rangle = 0 \end{aligned} \quad (3.12)$$

It is interesting to note that for the special choice of the constants  $\gamma_1 \gamma_2 = 1$  the two singlet states are identical. This means that any singlet can be reached from both trivial states

and there exists a one-to-one map between the two trivial solution spaces containing  $|s_1\rangle$  and  $|s_2\rangle$ . It is enough therefore to denote a singlet only by  $|s\rangle$  and keep in mind that it can be obtained from both  $|\varphi_i\rangle$ -s by different  $\gamma_i$ -s. The only time when the notation  $|s_i\rangle$  is useful is when one wants to explicitly show how one obtained that state.

An important question is under what conditions are the singlet states normalized. The normalization conditions led to some explicit values for the  $\gamma_i$ -s. To find the value of  $\gamma_1$  one can rewrite the annihilation operators in equations (3.11) in a somewhat simpler form by introducing the following operators:

$$\varphi(x) = -E^0(x) + i\gamma_1 E^\parallel(x) \quad (3.13)$$

$$\psi(x) = A^\parallel(x) - i\gamma_1 A^0(x) \quad (3.14)$$

$$\rho(x) = i\sqrt{-\Delta}\eta(x) + i\gamma_1 \overline{\mathcal{P}}(x) \quad (3.15)$$

$$k(x) = -\overline{\eta}(x) + \gamma_1 \frac{1}{\sqrt{-\Delta}} \mathcal{P}(x) \quad (3.16)$$

with the (anti)commutation relations:

$$[\varphi(x), \psi(y)] = 0 \quad (3.17)$$

$$[\varphi(x), \psi^\dagger(y)] = 2\gamma\delta^3(x-y) \quad (3.18)$$

$$[\rho(x), k(y)] = 0 \quad (3.19)$$

$$[\rho(x), k^\dagger(y)] = 2\gamma\delta^3(x-y) \quad (3.20)$$

The BRST-charge in this formulation is

$$Q = \frac{1}{2\gamma} \int d^3x [\varphi(x)\rho^\dagger(x) + \varphi^\dagger(x)\rho(x)] \quad (3.21)$$

while the singlet is going to be defined by:

$$\varphi(x) |s_1\rangle = \psi(x) |s_1\rangle = \rho(x) |s_1\rangle = k(x) |s_1\rangle = 0 \quad (3.22)$$

The simplest way to see whether a state given by the equation (3.11) is normalized or not is to use the wave-functional representation ( $\Psi[A, \eta, \overline{\eta}] \equiv \langle A, \eta, \overline{\eta} | s \rangle$ ). The first two equations in (3.11) are in this formulation:

$$(A^\parallel(x) - i\gamma A^0(x)) \Psi[A, \eta, \overline{\eta}] = 0 \quad (3.23)$$

In order to find a solution to the equation (3.23) one has to presume that the operators  $\hat{A}^0(x)$  have imaginary eigenvalues as mentioned in Section 2:

$$\hat{A}^0(x) | iA^0 \rangle = iA^0(x) | iA^0 \rangle \quad (3.24)$$

where  $A^0(x)$  is real. The formal inner product based on (2.6) gives the  $A$ -dependent functionals a norm:

$$\int \mathcal{D}^4 A \Psi^*[\vec{A}, -A^0] \Psi[\vec{A}, A^0] \quad (3.25)$$

The solution to equation (3.23) is given by

$$\Psi[A^\parallel, iA^0, \eta, \bar{\eta}] \propto \prod_x \delta(A^\parallel(x) + \gamma_1 A^0(x)) \quad (3.26)$$

Because of the imaginary eigenvalue of  $\hat{A}^0$  the operator equation corresponding to the second equation in (3.11) becomes:

$$\left( \gamma_1 \frac{\delta}{\delta A^\parallel(x)} - \frac{\delta}{\delta A^0(x)} \right) \Psi[A, \eta, \bar{\eta}] = 0 \quad (3.27)$$

We see that (3.26) is a solution to this equation too. The last two equations in (3.11) result in another proportionality:

$$\Psi[A^\parallel, iA^0, \eta, \bar{\eta}] \propto \prod_x \delta(\sqrt{-\Delta} \eta(x) + \gamma_1 \bar{\mathcal{P}}(x)) \quad (3.28)$$

(Note that here the fact that the ghost momenta are anti-hermitian eliminates the need for any other imaginary eigenvalues.) Computing now the norm of the vacuum state one obtains

$$\langle \Psi | \Psi \rangle = \prod_{\vec{x}} \frac{1}{\gamma_1} \gamma_1 = 1 \quad (3.29)$$

In this simple case one can understand this result in the following way: the wave-functional can be written as the product of a matter and a ghost part. The norm of the matter part is  $\prod_{\vec{x}} \frac{1}{2\gamma_1}$  while the norm of the ghost state is  $2\gamma_1$ . Starting from the other singlet one obtains the same result as expected. What we have found now is that every inner product vacuum state is normalized to unity. We also notice that in our case the trivial solutions could be written as products of a matter and a ghost state. Since the gauge regulator  $\exp(K_l)$  can also be decomposed in a matter and a ghost factor that commute with each other even the inner product singlet is a product of a matter and a ghost functional.

One can now find the orthogonal creation and annihilation operators that will eventually lead to the explicit form of the co-BRST operator. One needs to find those combinations of  $\varphi, \psi, \rho, k$  that give a positive and a negative definite matter (ghost) operator in every space point, that is combinations that are subject to the following (anti)commutation rules:

$$[D(x), D^\dagger(y)] = -[F(x), F^\dagger(y)] = \delta(x - y) \quad (3.30)$$

$$[D(x), F^\dagger(y)] = 0 \quad (3.31)$$

$$[G(x), G^\dagger(y)]_+ = -[H(x), H^\dagger(y)]_+ = \delta(x - y) \quad (3.32)$$

$$[G(x), H^\dagger(y)]_+ = 0 \quad (3.33)$$

where  $D(x)$  and  $F(x)$  are bosonic while  $G(x)$  and  $H(x)$  are fermionic annihilation operators. The coefficients in these linear combinations need not be constants, they can be coordinate dependent. If

$$D(x) = d_1(x)\varphi(x) + d_2(x)\psi(x) \quad (3.34)$$

$$F(x) = f_1(x)\varphi(x) + f_2(x)\psi(x) \quad (3.35)$$

$$G(x) = g_1(x)\rho(x) + g_2(x)k(x) \quad (3.36)$$

$$H(x) = h_1(x)\rho(x) + h_2(x)k(x) \quad (3.37)$$

then the (anti)commutation relations given before lead to:

$$|d_1(x)| = |f_1(x)| \quad |d_2(x)| = |f_2(x)| = \frac{1}{4\gamma |d_1(x)|} \quad (3.38)$$

$$|g_1(x)| = |h_1(x)| \quad |g_2(x)| = |h_2(x)| = \frac{1}{2 |h_1(x)|} \quad (3.39)$$

For notational simplicity from now on we do not explicitly denote the coordinate dependence of the operators. Restricting ourselves to real coefficients we obtain:

$$D = d_1\varphi + \frac{1}{4\gamma d_1}\psi \quad (3.40)$$

$$F = \pm(d_1\varphi - \frac{1}{4\gamma d_1}\psi) \quad (3.41)$$

$$G(x) = g_1\rho + \frac{1}{4\gamma g_1}k \quad (3.42)$$

$$H = \pm(g_1\rho - \frac{1}{4\gamma g_1}k) \quad (3.43)$$

It is clear that  $F^\dagger$  and  $H^\dagger$  create the negative normed states. There exists a class of metric operators that define new inner products such that every state is positive definite [4]:

$$\langle \xi | \hat{\eta} | \xi \rangle > 0, \forall | \xi \rangle \quad (3.44)$$

This metric operator is given by:

$$\hat{\eta} = e^{i\pi \int d^3x [F^\dagger F - H^\dagger H]} \quad (3.45)$$

The BRST operator in these variables for the positive sign solutions of (3.41) is

$$Q = \int d^3x \frac{1}{8\gamma d_1 g_1} [(D^\dagger + F^\dagger)(G + H) + (D + F)(G^\dagger + H^\dagger)] \quad (3.46)$$

The overall factor can always be chosen to be unity because it would not change the physical effect of the operator at any space point. The co-BRST is given by

$$^*Q \equiv \hat{\eta} Q \hat{\eta} = \int d^3x [(D^\dagger - F^\dagger)(G - H) + (D - F)(G^\dagger - H^\dagger)] \quad (3.47)$$

or expressed in the variables used in the definition of the singlets

$$^*Q = \int d^3x \frac{1}{\gamma} [\psi k^\dagger + \psi^\dagger k] \quad (3.48)$$

which is perfectly consistent with the definition of the singlet states as being the ones eliminated by  $\varphi, \psi, k, \rho$ . The same result is obtained for the negative sign solutions in (3.41) and (3.43). A similar expression was also found in [19]. Thus we come to the same conclusions as [4], namely that the singlet states are entirely defined by:

$$Q | s \rangle = ^*Q | s \rangle = 0 \quad (3.49)$$

Returning now to the original variables the co-BRST operator takes the form

$$^*Q = \int d^3x \left[ -\frac{1}{\gamma} A^\parallel \bar{\eta} + i\gamma A^0 \frac{1}{\sqrt{-\Delta}} \mathcal{P} \right] \quad (3.50)$$

We notice that for every choice of the coefficient  $\gamma$  there exists a co-BRST operator. In other words by fixing the co-BRST operator, that is by choosing  $\gamma$  we uniquely define the vacuum singlet state.

## 4 Generalized Gauge Fixing and Time Evolution

In the previous section we saw how the operators  $\exp(\gamma_i K_i)$  acting on the trivial BRST-invariant  $|\varphi_i\rangle$  states resulted in well defined inner product states denoted first as  $|s_i\rangle$ . We noticed then that every singlet state  $|s_i\rangle$  could be reached from both original  $|\varphi_i\rangle$ -s:

$$|s_i\rangle = \exp(\gamma_1 K_1) |\varphi_1\rangle = \exp\left(\frac{1}{\gamma_1} K_2\right) |\varphi_2\rangle \quad (4.1)$$

Since  $\gamma \neq 0$  and  $|\gamma| < \infty$  this equation means that the two states  $|\varphi_1\rangle$  and  $|\varphi_2\rangle$  are equivalent. The following question arises then: can we use instead of  $K_i$  a more general gauge-fixing function e.g. a linear combination of them? Defining now another operator by

$$K_3 \equiv \frac{i}{2}[K_1, K_2] = \frac{1}{2} \int d^3x [A^\parallel(x) E^\parallel(x) + A^0(x) E^0(x) + \mathcal{P}(x) \eta(x) - \bar{\eta}(x) \bar{\mathcal{P}}(x)] \quad (4.2)$$

one obtains an  $SL(2, \mathbb{R})$  commutation algebra:

$$[K_1, K_3] = -iK_1, \quad [K_2, K_3] = iK_2 \quad (4.3)$$

The standard form of the  $SL(2, \mathbb{R})$  algebra is easily recovered by introducing the linear combinations:  $X_1 = \frac{i}{2}(K_1 - K_2)$ ,  $X_2 = \frac{1}{2}(K_1 + K_2)$  and  $X_3 = K_3$ . It should be noted here that

$$K_1 |\varphi_2\rangle = K_2 |\varphi_1\rangle = K_3 |\varphi_1\rangle = K_3 |\varphi_2\rangle = 0 \quad (4.4)$$

leading to the possibility of using one common gauge-fixing operator for both  $|\varphi_i\rangle$ -s. This gauge-fixing operator is a linear combination of  $K_1$  and  $K_2$ . That is eq.(3.3) can be equivalently rewritten in the following way:

$$|s_l\rangle = e^{\alpha K_1 + \beta K_2} |\varphi_l\rangle \quad l = 1, 2 \quad (4.5)$$

The relation between the coefficients  $\alpha, \beta$  resp.  $\gamma_l$  in eq.(3.3) is given as in [3] by:

$$\gamma_1 = \frac{\alpha}{\sqrt{\alpha\beta}} \tanh \sqrt{\alpha\beta} \quad (4.6)$$

$$\gamma_2 = \frac{\beta}{\sqrt{\alpha\beta}} \tanh \sqrt{\alpha\beta} \quad (4.7)$$

These relations give some restrictions on the possible values of  $\alpha$  and  $\beta$ , namely that both have to be non-vanishing. If  $\alpha\beta < 0$  the  $\tanh$  goes of course over to  $\tan$ . What one notices



now is that there exists no such pair of finite  $\alpha, \beta$  such that it would lead to the same  $|\psi\rangle$  starting from the two different  $|\varphi\rangle$ .

So the answer on the question is yes, there is a large freedom in using a linear combination of the original gauge-fixing operators to obtain the same singlet. The next step in generalizing the gauge-fixing operator would be to add a term containing  $K_3$ . This would however only complicate the calculations without leading to physically new insights.

Nothing has been said yet about the **time evolution** of these various states although there seems to be a strong relationship between evolution and gauge fixing. The usual procedure to compute the time-evolution of any state is to act on it by a Hamiltonian operator defined in every space-time point:

$$\frac{d}{dt} |\psi(t)\rangle = i \int d^3x \mathcal{H}(\vec{x}, t) |\psi(t)\rangle \quad (4.8)$$

The corresponding integral equation used to be written as

$$|\psi(t)\rangle = e^{i \int_{t_0}^t dt' \int d^3x \mathcal{H}(\vec{x}, t')} |\psi(t_0)\rangle = e^{i \int_{t_0}^t dt' H(t')} |\psi(t_0)\rangle \quad (4.9)$$

The operator in the exponent can be written as as

$$\int_{t_0}^{\infty} dt' \theta(t - t') H(t') \quad (4.10)$$

Then using the formal relation

$$\theta(t - t') = \frac{1}{\partial_t} \delta(t - t') \quad (4.11)$$

one ends up with:

$$e^{i \frac{1}{\partial_t} H(t)} \quad (4.12)$$

It is easy to see that this operator when acting on  $|\psi(t_0)\rangle$  leads to the same differential equation (4.8). Let us see how this procedure applies to our case. The Hamiltonian obtained for the Lagrangean in equation (2.13) is:

$$H = \int d^3x \left[ \frac{1}{2} E^\mu E_\mu + \frac{1}{2} B^i B_i + E^0 \partial_i A^i + E^i \partial_i A_0 - i \bar{\mathcal{P}} \mathcal{P} + i \partial_i \bar{\eta} \partial^i \eta \right] \quad (4.13)$$

The first terms can be denoted as  $H_1 = \int d^3x [1/2 E^\mu E_\mu + 1/2 B^i B_i]$ , while the rest can be written as:

$$H_2 = [\tilde{\rho}_1 + \tilde{\rho}_2, Q] \quad (4.14)$$

where

$$\tilde{\rho}_1 = -i \int d^3x A^0 \mathcal{P} \quad (4.15)$$

$$\tilde{\rho}_2 = \int d^3x \partial_i A^i \bar{\eta} \quad (4.16)$$

and  $Q$  is of course the BRST-charge given in eq.(2.17). One can define an operator that on the equation of motion level is related to the time-evolution operator in (4.9):

$$U(t) = e^{i \int d^3x \frac{\mathcal{C}}{\sqrt{-\Delta}} \mathcal{H}(\vec{x}, t)} \quad (4.17)$$

where  $\mathcal{C}^{-1} = 4\pi \int_0^\infty dt (\frac{\pi}{t})^{\frac{3}{2}}$  is a renormalization constant. Let us take a closer look at this operator.

Inserting the Hamiltonian (4.13) into (4.17) one obtains:

$$U(t) = e^{iH'_1(t) + iH'_2(t)} \quad (4.18)$$

where

$$H'_1(t) \equiv \int d^3x \left( \frac{1}{2\sqrt{-\Delta}} E^\mu E_\mu + \frac{1}{4\sqrt{-\Delta}} F^{ij} F_{ij} \right) \quad (4.19)$$

and

$$H'_2(t) \equiv \int d^3x \left( E^0 A^\parallel - E^\parallel A_0 - i \frac{1}{\sqrt{-\Delta}} \bar{\mathcal{P}} \mathcal{P} + i \frac{\partial_i}{\sqrt{-\Delta}} \bar{\eta} \partial^i \eta \right) \equiv K_1 + K_2 \quad (4.20)$$

where  $K_1$  and  $K_2$  are the gauge-fixing operators we used before, i.e. the ones defined in equations (3.8) and (3.9). We notice now that  $H'_1$  and  $H'_2$  are members of a closed algebra.

The other members of this algebra are defined as follows:

$$H'_3 \equiv -\frac{1}{2} [H'_1, H'_2] = \int d^3x \frac{1}{(-\Delta)} E^0 E^\parallel \quad (4.21)$$

$$H'_4 \equiv [H'_2, H'_3] = \frac{1}{2} \int d^3x \left( \frac{1}{(-\Delta)} E^0 E_0 + \frac{1}{(-\Delta)} E^\parallel E_\parallel \right) \quad (4.22)$$

and the only remaining non-vanishing commutator is

$$[H'_2, H'_4] = 4H'_3 \quad (4.23)$$

The operator

$$H'_{ph} \equiv H'_1 - \frac{1}{2} H'_4 = \frac{1}{2} \int \left[ \frac{1}{\sqrt{-\Delta}} E^i E_i - \frac{1}{\sqrt{-\Delta}} E^\parallel E_\parallel + \frac{1}{\sqrt{-\Delta}} B^i B_i \right] \quad (4.24)$$

contains only the physical variables: the orthogonal components of the electric field and the magnetic field. Since it commutes with all the other operators one can write

$$U(t) = \exp(iH'_{ph}) \exp(iH'_2 + \frac{i}{2}H'_4) = \exp(iH'_{ph}) \exp(iK_1 + iK_2 + \frac{i}{2}H'_4) \quad (4.25)$$

We suppose now that the initial states  $|\varphi_i\rangle, i = 1, 2$ , defined in (3.1) and (3.2) evolve in time by this operator  $U(t)$ .

A particularly simple way to understand what this decomposition means is found if one goes over to an imaginary-time formalism by making the substitution  $it \rightarrow \tau$ . As shown in the Appendix the operator  $U(\tau)$  can further be decomposed when acting on any of the  $|\varphi_i\rangle$ -s. This decomposition is made possible by the fact that the operators in the exponent are members of an  $SL(2, R)$  algebra. For the first case one gets

$$\exp(K_1 + K_2 + \frac{1}{2}H'_4) |\varphi_1\rangle = \exp((\tanh 1)K_1) \exp\left(\frac{1}{2}(\tanh 1) \int d^3x \frac{1}{\sqrt{-\Delta}} E^\parallel E_\parallel\right) |\varphi_1\rangle \quad (4.26)$$

and one can define a new state:

$$|\varphi'_1\rangle \equiv e^{\frac{1}{2}(\tanh 1) \int \frac{1}{\sqrt{-\Delta}} E^\parallel E_\parallel} |\varphi_1\rangle \quad (4.27)$$

which is also a good trivial state since it is both BRST and anti-BRST invariant. We notice that this new state is not annihilated by all the constraints that define  $|\varphi_1\rangle$  in eq.(3.1). Instead of the first operator  $(\partial_i A^i)$  one has

$$[\partial_i A^i(x) + i(\tanh 1)E^\parallel(x)] |\varphi'_1\rangle = 0 \quad (4.28)$$

For  $|\varphi_2\rangle$  one gets a very similar result:

$$\begin{aligned} \exp(K_1 + K_2 + \frac{1}{2}H'_4) |\varphi_2\rangle &= \exp((\tanh 1)K_2) \exp\left(\frac{1}{2}(\tanh 1) \int \frac{1}{\sqrt{-\Delta}} E^0 E_0\right) |\varphi_2\rangle \equiv \\ &\equiv \exp((\tanh 1)K_2) |\varphi'_2\rangle \end{aligned} \quad (4.29)$$

where  $|\varphi'_2\rangle$  is again BRST and anti-BRST invariant and the operator that annihilates it instead of  $A^0$  in (3.2) is

$$[A^0(x) - i(\tanh 1) \frac{1}{\sqrt{-\Delta}} E^0(x)] |\varphi'_2\rangle = 0 \quad (4.30)$$

What we found is that when acting by the imaginary-time evolution operator on the original trivial  $|\varphi_l\rangle$  states  $U(\tau)$  could be decomposed as a product of three factors. One

of them  $\exp(\frac{1}{2} \tanh 1 \int \frac{1}{\sqrt{-\Delta}} E^\parallel E_\parallel)$  resp  $\exp(\frac{1}{2} \tanh 1 \int \frac{1}{\sqrt{-\Delta}} E^0 E_0)$  only rotated the  $|\varphi_l\rangle$  states into some new trivial states  $|\varphi'_l\rangle$  states. The second term is just a gauge regulation operator  $\exp((\tanh 1)K_l)$  and it transforms the trivial states into physical inner-product states  $|s_l\rangle$ . The remaining part of  $U(\tau)$  contains the physical Hamiltonian and it then generates the imaginary-time evolution of the inner product  $|s_l\rangle$  singlets.

$$|s(\tau)\rangle = e^{\int_{\tau_0}^{\tau} d\tau' H_{phys}} |s(\tau_0)\rangle \equiv e^{\int_{\tau_0}^{\tau} d\tau' \int d^3x \frac{1}{2} [E^i E_i - E^\parallel E^\parallel + B^i B_i]} |s(\tau_0)\rangle \quad (4.31)$$

All this suggests that even in the real-time case the evolution of the  $|s_l\rangle$  singlets is generated by the physical Hamiltonian (4.24). The other arguments that relate the time-evolution operator to the gauge-regulator are not valid in this case since the  $\exp(iK_l) |\varphi_l\rangle$  states are not inner product singlets, though there exist arguments in favour of using states like this in [20].

There exists another way too to understand these results if one defines the inner products as propagators in the path-integral formalism. See e.g. [21] for the case of supersymmetric particles, [22] for gravity and [23] for gravity in the Ashtekar variables. A more general picture is given in [24]. Here it is suggested that if the Hamiltonian in the classical theory is of the form  $H = H_{ph} + \{\rho, Q\}$ , the Hamilton operator in the quantum theory should be  $\hat{H} = \hat{H}_{ph} \pm i[\hat{\rho}, \hat{Q}]$ . Following this suggestion one could understand the decomposition in (4.25) the same way it was presented in the imaginary time case, namely: the unphysical  $([\rho, Q])$  part of the Hamilton operator transforms the original non-inner product states into physical inner product singlets and the physical part generates the time-evolution of these singlets.

Another thing to be noted is that the original definition of the  $|\varphi_l\rangle$  states in (3.1) and (3.2) is by no means unique. These trivial states were defined by the complete sets of operators eliminating them. What gives some freedom in the redefinition of these states is the fact that one of the operators (the  $\partial_i A^i$  for the first state and  $A^0$  for the second one) were gauge choices and we only expected them to complete a BRST doublet with the constraints. So one may always substitute the first operators ( $\partial_i A^i$  resp.  $A^0$ ) by a linear combination of themselves and the constraints since these linear combinations also satisfy the previous demand. This way one obtains new non-inner product states  $|\varphi'_l\rangle$ . One state in both classes defined in equations (4.28) and (4.30) have very nice “time-

evolution” properties. The corresponding part of the Hamiltonian can be substituted by the gauge-fixing operator  $K_I$ .

## 5 Final Remarks

One of the purposes of this paper was to try to find the relation between the co-BRST operator and the gauge-fixing operators that lead to well defined inner product states in electrodynamics. In [1] it was argued on general grounds that the trivial BRST-invariant states are not inner product states. In Section 2 we have explicitly proven this fact for the case of free electrodynamics. We found the two trivial solutions  $|\varphi_i\rangle, i = 1, 2$ . The inner products for both trivial solutions were shown to be ill-defined ( $\infty 0$ ).

Section 3 contains a treatment of the inner product problem more or less in the spirit of [3]. Starting from the two trivial BRST and anti-BRST invariant states ( $|\varphi_l\rangle$ ) one obtained the inner product singlets ( $|s_l\rangle$ ) by applying on them two gauge fixing regulators ( $\exp(\gamma_l[\rho_l, Q])$ ), where  $\gamma_l$  were free coefficients. It was shown that each singlet could be reached from both  $|\varphi_l\rangle$  states by a proper choice of the  $\gamma_l$  coefficients:

$$|s_\gamma\rangle = \exp(\gamma[\rho_1, Q]) |\varphi_1\rangle = \exp\left(\frac{1}{\gamma}[\rho_2, Q]\right) |\varphi_2\rangle \quad (5.1)$$

This means that at least in the case of Abelian models the two trivial solutions  $|\varphi_1\rangle$  and  $|\varphi_2\rangle$  are equivalent. The singlets are not only inner product states but their norm is even unity for any finite and non-vanishing  $\gamma_l$ . As expected from [1] the singlet states can be given as an orthogonal set, half of them being positive norm half of them negative norm states. After finding these states one could construct a metric operator ( $\hat{\eta}$ ) that would give positive norm to all the states. This metric operator lead to the co-BRST charge as described in [4]. We noticed that there is some freedom in the choice of the co-BRST operator arising from the freedom in choosing the orthogonal states. However once chosen the co-BRST operator together with the BRST operator completely define the inner product singlet states.

In Section 4 we made use of the fact that the gauge-fixing operators  $K_1$  and  $K_2$  belong to an  $SL(2, R)$  algebra. In this way one could define a more general gauge-fixing operator as the linear combination of the old ones:  $\exp(\alpha[\rho_1, Q] + \beta[\rho_2, Q])$ . It was shown that any

inner product singlet  $|s\rangle$  can be obtained this way and it was given the relation between  $\alpha, \beta$  resp.  $\gamma$  leading to the same singlet.

In the last part of Section 4 we analyzed and decomposed an interesting operator (4.17) which by some very formal arguments could be related to time evolution. From this decomposition we came to the conclusion that the time-evolution of the vacuum singlets in an imaginary-time formalism is governed by a physical Hamiltonian containing only the orthogonal components of the electric field and the magnetic field. The other terms in the Hamiltonian do not generate imaginary-time evolution. They are the gauge-fixing operators that transform the non-physical  $|\varphi_i\rangle$  states into the inner product  $|s\rangle$  states. A similar interpretation can be made even in the real time case if one follows the prescription in [24] in the construction of the Hamiltonian operator. The definition of the  $|\varphi_i\rangle$  states in (3.1) and (3.2) is not the most comfortable one from this point of view. These states were defined by giving a complete set of operators annihilating them. One of these operators in each case (the first ones in the definitions (3.1) and (3.2)) were in fact gauge choices and as such one can always change them and obtain some new non-inner product BRST-invariant states. It was shown that there exists a set of states  $|\varphi'_l\rangle$  defined in (4.28) and (4.30) such that the imaginary-time evolution operator acting on them was equivalent to a gauge regulator  $\exp(K_l)$ .

### Acknowledgment

I am very grateful to Robert Marnelius and to Igor A. Batalin for many useful discussions.

## Appendix

### Decomposition of the Hamiltonian

The operator acting on  $|\varphi_1\rangle$  in eq.(4.26) is

$$e^{i(K_1+K_2+\frac{1}{2}H_{41})} \quad (\text{A.1})$$

The terms in the exponent are members of an  $SL(2, R)$  algebra. To see this one can write  $H_4$  as a sum  $H_4 = H_{41} + H_{42}$ , where:

$$H_{41} = \frac{1}{2} \int \frac{1}{\sqrt{-\Delta}} E^0 E_0, \quad H_{42} = \frac{1}{2} \int \frac{1}{\sqrt{-\Delta}} E^\parallel E_\parallel \quad (\text{A.2})$$

Introducing the notation:

$$a = i(K_2 + H_{41}) \quad b = i(K_1 + H_{42}) \quad c = -2iK_3 \quad (\text{A.3})$$

the operator in eq.(A.1) becomes  $e^{a+b}$ . The algebra these operators satisfy is:

$$[a, b] = c, \quad [a, c] = 2a, \quad [b, c] = -2b \quad (\text{A.4})$$

Knowing that

$$a |\varphi_1\rangle = b |\varphi_2\rangle = c |\varphi_1\rangle = c |\varphi_2\rangle = 0 \quad (\text{A.5})$$

we can now compute how the operator in eq.(A.1) acts on the different initial states.

For case one we get:

$$\begin{aligned} e^{a+b} |\varphi_1\rangle &= [1 + b + \frac{1}{2!}b^2 + \frac{1}{3!}(b^3 + 2b) + \frac{1}{4!}(b^4 + 8b^2) + \\ &+ \frac{1}{5!}(b^5 + 20b^3 + 16b) + \frac{1}{6!}(b^6 + 40b^4 + 136b^2) + \dots] |\varphi_1\rangle = \\ &= e^{(\tan 1)b} |\varphi_1\rangle = e^{i(\tan 1)(K_1+H_{41})} |\varphi_1\rangle \end{aligned} \quad (\text{A.6})$$

Since  $K_1$  and  $H_{41}$  commute with each other one can write

$$e^{(K_1+K_2+\frac{1}{2}H_4)} |\varphi_1\rangle = e^{(\tanh 1)K_1} e^{(\tanh 1)H_{41}} |\varphi_1\rangle \quad (\text{A.7})$$

The treatment of the case 2 is trivially similar to the first case.

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